Exact vectorial law for homogeneous rotating turbulence

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Three-dimensional hydrodynamic turbulence is investigated under the assumptions of homogeneity and weak axisymmetry. Following the kinematics developed by E. Lindborg [J. Fluid Mech. **302**, 179 (1995)] we rewrite the von Kármán–Howarth equation in terms of measurable correlations and derive the exact relation associated with the flux conservation. This relation is then analyzed in the particular case of turbulence subject to solid-body rotation. We make the ansatz that the development of anisotropy implies an algebraic relation between the axial and the radial components of the separation vector **r** and we derive an exact vectorial law which is parametrized by the intensity of anisotropy. A simple dimensional analysis allows us to fix this parameter and find a unique expression.

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I. INTRODUCTION

Among the great unsolved problems in classical physics turbulence is certainly the most challenging one which has evaded a general physical understanding for many decades. Therefore, any exact laws derived in turbulence appear extremely important. In his third 1941 paper, Kolmogorov found that an exact and nontrivial relation may be derived for Navier-Stokes equations in terms of third-order longitudinal structure function [1]. Because of the rarity of such results, the Kolmogorov's four-fifths law is considered as one of the most important results in turbulence [2]. The derivation of the Kolmogorov's law uses earlier exact relations found by von Kármán and Howarth in 1938: it is the well-known von Kármán–Howarth (vKH) equation that describes the dynamical evolution of the second-order correlation tensors [3]. Few extensions of such results (vKH equations and four-fifths law) to other fluids have been made; it concerns, for example, scalar passively advected [4] such as the temperature or a pollutant in the atmosphere, quasigeostrophic flows [5] and astrophysical magnetized fluid described in the framework of magnetohydrodynamics (MHD) [6,7], electron and Hall MHD [8,9].

In this paper, I investigate the problem of homogeneous, weak axisymmetric, incompressible, three-dimensional (3D) hydrodynamic turbulence with an application to rotating flows. The kinematics of axisymmetric turbulence has been analyzed in the past first by Batchelor (1946) [10] and more widely by Chandrasekhar (1950) [11]. However, the formalism proposed by Chandrasekhar based on skew tensors did not allow to solve the vKH equation nor to express it with measurable correlations. A new formalism for the representation of two-point correlation tensors has been introduced by Lindborg (1995) [12] but only the kinematics was discussed not the dynamics. The present paper is mainly devoted to the dynamics of such a system. We first rewrite the vKH equation in terms of measurable correlations and derive the exact relation associated with energy. This relation is then analyzed in the particular case of rotating turbulence with Ω the rotating rate. We know that under rotation turbulence develops anisotropy with a nonlinear transfer mainly in the direction transverse to Ω [13]. By considering this property we make the ansatz that the development of anisotropy implies an algebraic relation between the axial and the radial components of the separation vector **r**. After performing an integration over a class of manifolds we obtain an exact vectorial law which is parametrized by the intensity of anisotropy. A simple dimensional analysis allows us to fix this parameter and find a unique exact expression. The vectorial law generalizes the well-known four-fifths law found by Kolmogorov which implies only scalars.

The organization of the paper is as follows. Section II is devoted to the kinematics of axisymmetric turbulence: we remind the basic steps whereas some details are given in the Appendix; we write the exact relation of flux conservation in terms of measurable correlation tensors. In Sec. III we introduce structure functions and in Sec. IV we derive the exact vectorial law. A discussion and conclusion is given in Sec. V.

II. HOMOGENEOUS AXISYMMETRIC TURBULENCE

A. Homogeneous turbulence

The well-known exact relation—the four-fifths law—found by Kolmogorov (1941) for Navier-Stokes turbulence is only valid when isotropy is assumed. Without this assumption, it is possible to write the following relation [2,14]:

$$-\frac{1}{4}\nabla_{\mathbf{r}}\cdot\mathbf{F}(\mathbf{r})=\varepsilon, \tag{1}$$

where ε is the mean energy dissipation rate per unit mass and

$$\mathbf{F}(\mathbf{r})=\langle\delta\mathbf{v}\,\delta v_i^2\rangle, \tag{2}$$

with $\delta\mathbf{v}=\mathbf{v}(\mathbf{x}+\mathbf{r})-\mathbf{v}(\mathbf{x})$. It is important to realize that (i) relation (1) is nothing else than the expression of the energy flux conservation in the inertial range; (ii) it is straightforward to prove that expression (2) satisfies the Navier-Stokes equations when homogeneity is assumed (and not necessarily isotropy); (iii) there is a big difference between the four-fifths law found by Kolmogorov and expressions (1) and (2) since the former is a proper mathematical solution of the

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latter (when isotropy is assumed). Actually, relation (1) is a condition that the homogeneous solutions must satisfy [2]. For that reason it is preferable to reserve the term "four-fifths law" only for the isotropic relation which may be written as [15]

$$-\frac{4}{3}\varepsilon r = \langle \delta v_{\parallel} \delta v_i^2 \rangle, \tag{3}$$

or more classically as [1]

$$-\frac{4}{5}\varepsilon r = \langle \delta v_{\parallel}^3 \rangle, \tag{4}$$

where the symbol $\parallel$ means the (longitudinal) direction along the vector separation $\mathbf{r}$. To date no exact law has been derived in the general case of homogeneous anisotropic turbulence. It is basically the goal of this paper to demonstrate that such an exact law may be derived in the specific case of axisymmetry. In this case the law has a vectorial nature which is linked to the anisotropy of the problem.

B. Axisymmetric turbulence

The fundamental difference between isotropic and axisymmetric turbulence is the existence in the latter case of a privileged direction $\boldsymbol{\lambda}$. A turbulent flow is said axisymmetric when it is axially symmetric about the (unit) vector $\boldsymbol{\lambda}$. We may also consider statistical correlations invariant for reflections in planes containing $\boldsymbol{\lambda}$ and perpendicular to $\boldsymbol{\lambda}$. When only the latter invariance is satisfied the axisymmetry is said weak whereas if both are satisfied the axisymmetry is said strong. In the present paper we will consider the most general case of weak axisymmetry which allows us to have helical terms. Note that the helical terms involve fields which are not dyadic because of the nonconservation of the mirror symmetry. Previous works [10,11] devoted to axisymmetric turbulence showed that the correlation tensors may be expressible in terms of the fundamental invariants associated to this group of symmetry. In mathematical terms, that means we may build a two-point correlation tensor of an arbitrary order, $F(\boldsymbol{\lambda},\mathbf{r};\mathbf{a},\mathbf{b},\ldots)$, which is invariant under an arbitrary rotation or reflection of the vector configuration formed by $\mathbf{r},\boldsymbol{\lambda},\mathbf{a},\mathbf{b},\ldots$ with $\mathbf{r}$ as the vector separation between the two points of measurement.

Despite the mathematical elegance of Chandrasekhar's representation [11] in which skew tensors are used to satisfy implicitly the divergence free condition, the final writing of the equivalent of the vKH equations for axisymmetric Navier-Stokes flows remains formal and no explicit relation in terms of measurable quantities is possible to obtain. In this paper, we follow the representation proposed by Lindborg for axisymmetric Navier-Stokes fluids and show that this kinematics may lead to a simplification of relation (1). This new expression will be the point of departure for deriving an exact vectorial law for rotating turbulence.

The nth-order two-point correlation tensor of homogeneous axisymmetric turbulence can be represented by a sum of all linearly independent nth-order tensors that can be formed from the separation vector $\mathbf{r}$ between the two measurement points (denoted M and M') in the turbulent field, the axial vector $\boldsymbol{\lambda}$ and the Kronecker tensor [16]. Each of the tensors multiplies a scalar which is only a function of the length r and the angle between $\mathbf{r}$ and $\boldsymbol{\lambda}$. The definition of nth-order two-point correlation tensors is recalled in the Appendix. One of the most convenient representations for these tensors implies the following orthogonal unit vectors [12]:

$$\boldsymbol{\lambda}, \tag{5}$$

$$\mathbf{e}^{(1)} = \frac{1}{\rho}\boldsymbol{\lambda} \times \mathbf{r}, \tag{6}$$

$$\mathbf{e}^{(2)} = \mathbf{e}^{(1)} \times \boldsymbol{\lambda}, \tag{7}$$

where $\rho=|\mathbf{r}\times\boldsymbol{\lambda}|$. Note that the orthonormal basis $(\boldsymbol{\lambda},\mathbf{e}^{(1)},\mathbf{e}^{(2)})$ does not imply directly the vector $\mathbf{r}$ but a vector product of $\mathbf{r}$. Such a decomposition leads to a significant simplification in the interpretation of the vKH equations. It is important to note that this decomposition is not adapted to the situation where $\mathbf{r}$ is parallel to $\boldsymbol{\lambda}$, whereas the case $\mathbf{r}$ perpendicular to $\boldsymbol{\lambda}$ does not raise any problem. The decomposition used is not unique: for example, we may also use the polar-spherical coordinates from which the longitudinal (along $\mathbf{r}$) and transverse (perpendicular to $\mathbf{r}$) field components directly appear. However—as it will be shown in the second part of the paper—these components are not the most important for this axisymmetric problem.

It is interesting to note that the previous decomposition for axisymmetric turbulence may also be considered in Fourier space where a complete theory exists [13,17]. Following a general decomposition of the second-order spectral tensor in terms of energy, polarization, and helicity, it is possible to find generalized Lin equations which may incorporate different effects such as rotation, stratification, or distortion. With such a decomposition the divergence free condition is automatically satisfied (the field is said dyadic) and the pressure-velocity correlations implicitly solved [17].

C. Dynamical equations

The starting point of our analysis is the 3D Navier-Stokes equations with the Coriolis force

$$\partial_t \mathbf{v} + \mathbf{f} = -\boldsymbol{\nabla} P - \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} + \nu\Delta\mathbf{v}, \tag{8}$$

$$\boldsymbol{\nabla}\cdot\mathbf{v} = 0, \tag{9}$$

where $\mathbf{v}$ is the flow velocity, $\mathbf{f}=2\boldsymbol{\Omega}\times\mathbf{v}$ the Coriolis force with Ω as the rotating rate, P is the pressure, and ν is the viscosity. One objective is the derivation of the equivalent of Eq. (1) for homogeneous axisymmetric turbulence. We will use the kinematics proposed by Lindborg [12] and rewrite the vKH equations. Useful information about the kinematics are recalled in the Appendix. From Eq. (8) we obtain

$$\begin{aligned}\partial_t R_{ij}(\mathbf{r}) &= \partial_t\langle v_i v_j'\rangle \\ &= \langle v_i \partial_t v_j'\rangle + \langle v_j' \partial_t v_i\rangle \\ &= -\langle v_i f_j'\rangle - \langle v_j' f_i\rangle - \langle v_i v_\ell' \partial_\ell' v_j'\rangle - \langle v_i \partial_j' P'\rangle\end{aligned}$$

$$-\langle v_j' v_\ell \partial_\ell v_i\rangle - \langle v_j' \partial_i P\rangle + \nu\langle v_i \partial_{\ell\ell}'^2 v_j'\rangle + \nu\langle v_j' \partial_{\ell\ell}^2 v_i\rangle. \tag{10}$$

After simple manipulations where we use, in particular, the divergence free condition and the homogeneity assumption, we get

$$\partial_t R_{ij}(\mathbf{r}) = \partial_{r_\ell}[S_{i\ell j}(\mathbf{r}) - S_{j\ell i}(-\mathbf{r})] - \langle v_i f_j'\rangle - \langle v_j' f_i\rangle + \partial_{r_i} P_j(\mathbf{r}) - \partial_{r_j} P_i(-\mathbf{r}) + 2\nu\partial^2_{r_\ell r_\ell} R_{ij}(\mathbf{r}), \tag{11}$$

where $S_{i\ell j}(\mathbf{r})=\langle v_i v_\ell v_j'\rangle$ and $P_i(\mathbf{r})=\langle P v_i'\rangle$. Contrary to the isotropic case, in axisymmetric turbulence the velocity-pressure correlation terms contribute to the energy transfer between the radial and axial fluctuations, i.e., the nondiagonal part of the second-order correlation tensor. It also contributes to the different terms of the diagonal part with a global contribution equal to zero. Then the diagonal part of these equations gives

$$\partial_t R_{ii} = 2\partial_{r_\ell} S_{i\ell i} + 2\nu\partial^2_{r_\ell r_\ell} R_{ii}. \tag{12}$$

This equation is apparently similar to the isotropic case since the contribution of the Coriolis force does not appear because this force produces no work. However, because of the axisymmetric nature of the problem the expressions of the correlation tensors (see the kinematics) are in fact fundamentally different. Note that the counterpart of expression (12) in Fourier space exists: it is the generalized Lin equation [17].

D. vKH equations and flux conservation

The kinematics for axisymmetric turbulence can now be incorporated in the dynamical Eq. (12) which gives [with relations (A4)–(A6)] the following vKH equation:

$$\frac{1}{2}\partial_t R_{ii} = \frac{1}{\rho}\frac{\partial}{\partial\rho}[\rho(S_5 + S_7 + S_{10})] + \frac{\partial}{\partial z}[S_1 + S_8 + S_9] + \nu\partial^2_{r_\ell r_\ell} R_{ii}. \tag{13}$$

Exact solution of axisymmetric turbulence may be derived from Eq. (13) for third-order correlation tensors by assuming the following assumptions specific to fully developed turbulence [2]. First, we consider the long time limit for which a stationary state is reached with a finite mean energy rate per unit mass. Second, we take the infinite Reynolds number limit ($\nu \to 0$) for which the mean energy dissipation rate per unit mass tends to a finite positive limit, ε. Under these assumptions, it is straightforward to show (by including structure functions) that

$$\frac{1}{2}\partial_t R_{ii} = -\varepsilon. \tag{14}$$

Finally, we obtain the following exact relation in the inertial range:

$$-\varepsilon = \frac{1}{\rho}\frac{\partial}{\partial\rho}[\rho(S_5 + S_7 + S_{10})] + \frac{\partial}{\partial z}[S_1 + S_8 + S_9] = \boldsymbol{\nabla}_{\mathbf{r}} \cdot \begin{pmatrix} S_5 + S_7 + S_{10} \\ 0 \\ S_1 + S_8 + S_9 \end{pmatrix}. \tag{15}$$

Note that the second expression is trivially obtained by using the definition of the divergence operator in cylindrical coordinates (the zero in the second line simply means that the problem is axisymmetric). This expression means that we have the conservation of energy flux through the inertial range. At this level of analysis, its solution is still highly nontrivial.

Contrary to previous works [10–12], the new form of vKH equation will be written in terms of measurable correlation tensors. More precisely, if we define the vector fields

$$\mathbf{v} = u\boldsymbol{\lambda} + v\mathbf{e}^{(2)} + w\mathbf{e}^{(1)}, \tag{16}$$

we obtain by definition the following third-order correlation tensors:

$$S_1 = \langle u^2 u'\rangle, \tag{17}$$

$$S_5 = \langle v^2 v'\rangle, \tag{18}$$

$$S_7 = \langle uvu'\rangle, \tag{19}$$

$$S_8 = \langle uvv'\rangle, \tag{20}$$

$$S_9 = \langle uww'\rangle, \tag{21}$$

$$S_{10} = \langle vww'\rangle. \tag{22}$$

The substitution of these expressions into Eq. (15) gives

$$-\varepsilon = \boldsymbol{\nabla}_{\mathbf{r}} \cdot \begin{pmatrix} \langle v^2 v' + uvu' + vww'\rangle \\ 0 \\ \langle u^2 u' + uvv' + uww'\rangle \end{pmatrix}. \tag{23}$$

III. EXACT RELATION FOR AXISYMMETRIC TURBULENCE

To find a solution of our problem, we first need to rewrite expression (23); after some manipulations, we obtain the following simple expressions:

$$-\varepsilon = \boldsymbol{\nabla}_{\mathbf{r}} \cdot \begin{pmatrix} \langle v(\mathbf{v}\cdot\mathbf{v}')\rangle \\ 0 \\ \langle u(\mathbf{v}\cdot\mathbf{v}')\rangle \end{pmatrix} = \boldsymbol{\nabla}_{\mathbf{r}} \cdot \langle \mathbf{v}_P(\mathbf{v}\cdot\mathbf{v}')\rangle, \tag{24}$$

where by definition $\mathbf{v}_P$ is the projection of the velocity in the $\boldsymbol{\lambda}-\mathbf{e}^{(2)}$ plane. We now introduce structure functions and write

$$\langle \delta\mathbf{v}_P \delta v_i^2\rangle = \langle(\mathbf{v}_P' - \mathbf{v}_P)(\mathbf{v}^2 + \mathbf{v}'^2 - 2\mathbf{v}\cdot\mathbf{v}'\ \rangle = -\tilde{\mathbf{P}}(\mathbf{r}) + \tilde{\mathbf{P}}(-\mathbf{r}) + 2\langle(\mathbf{v}_P - \mathbf{v}_P')\mathbf{v}\cdot\mathbf{v}'\rangle, \tag{25}$$

with the first-order correlation tensor $\tilde{\mathbf{P}}(\mathbf{r}) = \langle \mathbf{v}_P \mathbf{v}'^2 \rangle$. We note the following simplification when the divergence operator is applied:

$$\nabla_{\mathbf{r}} \cdot \langle \delta \mathbf{v}_P \delta v_i^2 \rangle = 4 \nabla_{\mathbf{r}} \cdot \langle \mathbf{v}_P (\mathbf{v} \cdot \mathbf{v}') \rangle, \tag{26}$$

which finally gives the new expression

$$-4\varepsilon = \nabla_{\mathbf{r}} \cdot \langle \delta \mathbf{v}_P \delta v_i^2 \rangle. \tag{27}$$

This result is in agreement with previous finding dedicated to the general case of 3D (without rotation) turbulence [14] as explained in Sec. II A. The reason is that the Coriolis force introduced in Eq. (8) has no effect in the subsequent derivation since it produces no work; therefore, expression (27) may be seen as a general relation for axisymmetric Navier-Stokes turbulence. Note that since the problem is axisymmetric we only have two degrees of freedom and thus vectors projected into the $\boldsymbol{\lambda} - \mathbf{e}^{(2)}$ plane.

IV. ROTATING TURBULENCE

A. Vectorial nature of the relation

It is well known that rotation introduces anisotropy at every scale with a reduction in transfer along the $\boldsymbol{\Omega}$ direction (which corresponds to $\boldsymbol{\lambda}$ in our formalism). Direct evidences have been found in experiments [18,19], direct numerical simulations [20,21], closure models [22], and wave turbulence [23,24]. Some nonlinear features are accurately characterized through Fourier space analyses with no equivalent in physical space using the Kolmogorov (2D or 3D) laws. It is the case with the detailed (triad by triad) conservation of invariants such as energy and kinetic helicity (in 3D) or enstrophy (in 2D); the regime of asymptotically fast rotation which corresponds to wave turbulence gives another example where we may derive wave kinetic equations for invariants as well as their exact power-law solutions [23]. It is important to stress that rapid rotation does not necessarily imply conventional two-dimensional state and inverse energy cascade as illustrated by wave turbulence. It is also interesting to note that some direct numerical simulations [20] do not support a completely conventional two-dimensionalization process.

The vectorial nature of relation (27) is important information that has to be correctly interpreted. It would be a mistake to think that for any fixed direction of the separation vector $\mathbf{r}$—i.e., a given angle between $\mathbf{r}$ and $\boldsymbol{\lambda}$—the exact relation is well satisfied for a wide range of distance r. In particular, any measurement far from the radial direction, and especially close to the (axial) $\boldsymbol{\lambda}$ direction, might give a different behavior (we remind that the formalism collapses for $\mathbf{r}$ exactly parallel to $\boldsymbol{\lambda}$). Indeed, when the rotating rate is large the correlation is mostly effective at large angle with respect to $\boldsymbol{\lambda}$ which corresponds to a nonlinear transfer mainly in the radial direction. Note that under the hypothesis of slow rotation, a first attempt has been made recently to evaluate the approximate form of the exact law—a modification of the Kolmogorov's four-fifths law—and to understand the role of a possible inverse cascade [25].

B. 3D and 2D isotropic limits

Before going further in the analysis it is important to first remind the situation for 3D and 2D isotropic turbulence. For 3D isotropic turbulence, the exact relation (27) may be integrated over a ball (a full sphere) of radius r since then the $\boldsymbol{\lambda}$ direction does not play any particular role. Simple calculation gives

$$-4\varepsilon \int\int\int_{ball} d\mathcal{V} = \int\int\int_{ball} \nabla_{\mathbf{r}} \cdot \langle \delta \mathbf{v}_P \delta v_i^2 \rangle d\mathcal{V},$$

$$-\frac{16\pi r^3}{3}\varepsilon = \int\int_{sphere} \langle \delta \mathbf{v}_P \delta v_i^2 \rangle \cdot d\mathcal{S},$$

$$-\frac{4}{3}\varepsilon r = \langle \delta v_L \delta v_i^2 \rangle, \tag{28}$$

where L means—as usual—the component along the $\mathbf{r}$ direction.

For 2D turbulence, the vector fields are only 2D (without λ component) and the vector separation $\mathbf{r}$ is taken inside a disk perpendicular to the $\boldsymbol{\lambda}$ direction. Then, we obtain trivially (by taking the opposite sign since an inverse cascade is expected)

$$4\varepsilon \int\int_{disk} d\mathcal{S} = \int\int_{disk} \nabla_{\mathbf{r}} \cdot \langle \delta \mathbf{v}_P \delta v_i^2 \rangle d\mathcal{S},$$

$$4\pi r^2 \varepsilon = \int_{circle} \langle \delta \mathbf{v}_P \delta v_i^2 \rangle d\mathcal{L},$$

$$2\varepsilon r = \langle \delta v_L \delta v_i^2 \rangle. \tag{29}$$

We see that in both cases we recover the well-known limits [1,26]. It is important to understand what we have done implicitly by performing an integration over a ball or a disk: we have assumed that the nonlinear transfers are 3D or 2D without any preferential direction. For anisotropic turbulence the nonlinear transfer has a preferential direction that should be incorporated when the integration is performed. It is basically the goal of the next section to evaluate this manifold.

C. Exact vectorial law for a class of manifolds

We arrive now at the heart of the problem which consists in solving Eq. (27) for rotating turbulence. The resolution of this equation is not obvious in the most general case since one needs to find a volume such that the flux is constant at its surface. Then, one can perform an integration of Eq. (27) over this volume, apply the Stoke's theorem, and obtain a simple expression. We believe that the existence of such a volume is still an open question. However, the problem that we are dealing with is particular. Indeed, the divergence operator implies the vector $\mathbf{r}$ which is the vector separation between the two points of measurement. We have seen in the previous section that the vector $\mathbf{r}$ follows a preferential direction when the external agent $\boldsymbol{\Omega}$ is present. In particular,

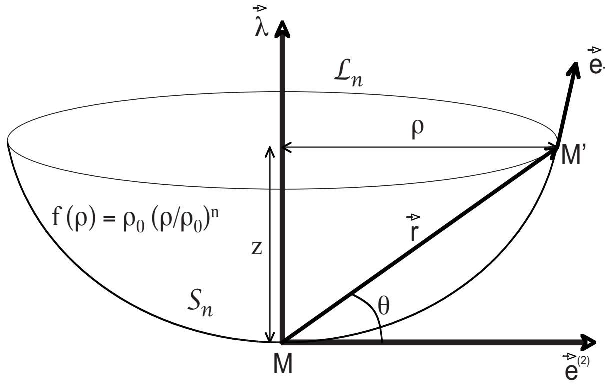

FIG. 1. For axisymmetric turbulence, we perform an integration of Eq. (27) over a manifold $\mathcal{S}_n$ defined by the function $f_n(\rho)=\rho_0(\rho/\rho_0)^n$. In order to take into account the development of anisotropy at small scales ($\mathbf{r}$ close to $\mathbf{e}^{(2)}$ for small r) the exponent n has to be greater than 1.

we know that anisotropy is stronger at small scales than at large scales. In terms of correlations that means we expect to get correlations with a vector $\mathbf{r}$ close to the transverse direction when r is taken small. This property clearly means a relationship between the axial and the radial components of $\mathbf{r}$. Following this idea we shall integrate the exact relation (27) over an axisymmetric manifold $\mathcal{S}_n$ defined by the function (see Fig. 1)

$$z = f_n(\rho) = \rho_0\left(\frac{\rho}{\rho_0}\right)^n. \quad (30)$$

It is the simplest algebraic function satisfying the conditions $f_n(\rho)\to 0$ when $\rho\to 0$ with a simple power-law dependence between ρ and z. Other (exponential or logarithmic) functions may lead to a more complex form with possible trouble to satisfy the previous conditions. Without loss of generality we may already note that n must be greater than 1 to satisfy the anisotropic property at small scales ($\mathbf{r}$ close to $\mathbf{e}^{(2)}$). Finally, note that ρ_0 is the value of ρ for which the angle between $\mathbf{r}$ and $\boldsymbol{\lambda}$ is $\pi/4$; therefore, ρ/ρ_0 may be seen as a way to delimit the space into two domains where the correlation vector is closer to the radial or to the axial direction.

Now we perform an integration of the exact relation (27) over $\mathcal{S}_n$ and get

$$-4\varepsilon \int\int d\mathcal{S}_n = \int\int \boldsymbol{\nabla}_{\mathbf{r}}\cdot\langle\delta\mathbf{v}_P\delta v_i^2\rangle d\mathcal{S}_n,$$

$$-4\varepsilon\mathcal{S}_n = \int_{circle}\langle\delta\mathbf{v}_P\delta v_i^2\rangle d\mathcal{L}_n, \quad (31)$$

which gives the exact law

$$-\frac{4\varepsilon\mathcal{S}_n}{2\pi\rho} = \langle\delta v_T\delta v_i^2\rangle, \quad (32)$$

where by definition T means the tangent direction at point M' (see Fig. 1). If we introduce the unit vector $\mathbf{e_T}$ along that direction we obtain the vectorial relation

$$-\frac{2\varepsilon\mathcal{S}_n}{\pi\rho}\mathbf{e_T} = \langle\delta\mathbf{v}_T\delta v_i^2\rangle, \quad (33)$$

with

$$\mathbf{e_T} = \frac{\mathbf{e}^{(2)} + f_n'(\rho)\boldsymbol{\lambda}}{\sqrt{1+f_n'(\rho)^2}} = \frac{\mathbf{e}^{(2)} + n(\rho/\rho_0)^{n-1}\boldsymbol{\lambda}}{\sqrt{1+n^2(\rho/\rho_0)^{2(n-1)}}} = \frac{\mathbf{e}^{(2)} + n\tan\theta\boldsymbol{\lambda}}{\sqrt{1+n^2\tan^2\theta}}, \quad (34)$$

where θ is the angle between $\mathbf{r}$ and $\mathbf{e}^{(2)}$ (see Fig. 1). The surface $\mathcal{S}_n$ for a given ρ is defined as

$$\mathcal{S}_n = \int 2\pi\rho d\ell = \int_0^\rho 2\pi\rho\sqrt{1+f_n'(\rho)^2}d\rho$$
$$= \int_0^\rho 2\pi\rho\sqrt{1+n^2\left(\frac{\rho}{\rho_0}\right)^{2(n-1)}}d\rho$$
$$= \frac{\pi\rho_0^2}{n^{2/(n-1)}}\int_0^X \sqrt{1+X^{n-1}}dX, \quad (35)$$

with

$$X = n^{2/(n-1)}\left(\frac{\rho}{\rho_0}\right)^2 = \left(\frac{nz}{\rho}\right)^{2/(n-1)} = (n\tan\theta)^{2/(n-1)}. \quad (36)$$

The combination of the different expressions gives eventually the following exact vectorial law:

$$-2\frac{I(X)}{X}\varepsilon\rho\mathbf{e_T} = \langle\delta\mathbf{v}_T\delta v_i^2\rangle, \quad (37)$$

where

$$I(X) = \int_0^X \sqrt{1+X^{n-1}}dX. \quad (38)$$

It is the first important result of the paper. We first note that relation (37) is close to the one derived when isotropy is assumed [1]. The difference resides in the prefactor that has to be computed and in the T direction. In the next section we will go further in the analysis by fixing the parameter n which measures the intensity of anisotropy. Second, from relation (36) we see that X may be considered as a measure of the θ angle along which the correlation is made. The direct consequence is that the exact vectorial law is not as universal as in the isotropic case since the prefactor depends on the angle. However, if the correlation is measure at a fixed θ angle then the prefactor is fixed and the law keeps a degree of universality. Third, this general law depends on two variables (ρ and θ) which means that the surface $\mathcal{S}_n$ (see Fig. 1) has to be seen as a particular representation of the surface integral: all the correlation space may be filled by taking eventually any values of ρ and θ.

D. Critical balance

The exact vectorial relation (37) implies a parameter n that has to be determined. It is the purpose of this section to fix n by a dimensional analysis based on the critical balance idea [27,28]. Before that, we may guess the general behavior of our system. According to the discussion of Sec. IV A, we expect that both the intensity of Ω and the distance r will have an influence on the direction of $\mathbf{r}$. For a given distance r we expect that the direction $\mathbf{r}$ will be close to $\mathbf{e}^{(2)}$ for large Ω. In the meantime, for a given Ω we expect to measure stronger anisotropy at smaller scale and therefore as one goes at small r, the vector $\mathbf{r}$ is expected to have a direction closer to $\mathbf{e}^{(2)}$. We will see that this qualitative analysis is sustained by a quantitative (dimensional) analysis.

To investigate further this idea we shall restrict our analysis to the inviscid, stationary Navier-Stokes equations (written for the vorticity) since basically we want an interpretation of the exact solution valid in the inertial range; we thus obtain

$$-2(\mathbf{\Omega}\cdot\mathbf{\nabla})\mathbf{v}=(\mathbf{w}\cdot\mathbf{\nabla})\mathbf{v}-(\mathbf{v}\cdot\mathbf{\nabla})\mathbf{w}. \tag{39}$$

By noting that the terms in the right-hand side are of the same order, we arrive at the nontrivial balance

$$\Omega\partial_z \sim w_r \mathbf{\nabla}_r, \tag{40}$$

which may also be written as

$$\frac{w_r}{\Omega} \sim \frac{\partial_z}{\mathbf{\nabla}_r} = \frac{k_z}{k_r} = \sin\,\theta, \tag{41}$$

where θ is also the angle between the separation vector $\mathbf{r}$ and $\mathbf{e}^{(2)}$ (see Fig. 1). As we see, relation (41) offers a direct evaluation of the $\mathbf{r}$ direction: therefore, although the rotating rate does not enter explicitly in vectorial law (37), it constrains the direction along which the scaling law applies.

To pursue the analysis one needs to evaluate the scale dependence of w_r. A naive evaluation would consist to use the relation $w_r \sim v_r/r$ by claiming that the distance which enters into account is also the separation distance. But as we know from isotropic turbulence, helical flows satisfy a nontrivial exact relation [29,30] which corresponds dimensionally to

$$w_r \sim r^{1/3}. \tag{42}$$

This scaling law leads dimensionally to a kinetic helicity spectrum in $k^{-5/3}$ (with $v_r \sim r^{1/3}$) which is compatible with direct numerical simulation [31], whereas the previous naive scaling leads to a less steep power law in $k^{-2/3}$ which has never been observed. We make the choice to follow scaling (42) [32]; then from relation (41) we obtain

$$\sin\,\theta \sim \frac{r^{1/3}}{\Omega}. \tag{43}$$

In other words, this expression means that the scaling relation depends on the strength of the rotating rate (as explained above), with an orientation closer to $\mathbf{e}^{(2)}$ for a strong Ω, but also on the scales itself with a direction getting closer to $\mathbf{e}^{(2)}$ at small scales (smaller r). This dimensional analysis will be used in next section to derive the vectorial law for rotating anisotropic turbulence since relation (43) gives the following dimensional small-scale constrain:

$$\sin\,\theta \sim \frac{\rho^{1/3}}{\Omega} \sim \frac{z}{\rho} = \left(\frac{\rho}{\rho_0}\right)^{n-1}, \tag{44}$$

which leads to

$$n = 4/3. \tag{45}$$

Note that we also have $\rho_0 \sim \Omega^3$.

E. Exact vectorial law for rotating turbulence

Following the critical balance idea we shall rewrite expression (37) for $n=4/3$ which gives

$$-2f(\theta)\varepsilon\rho\mathbf{e_T} = \langle \delta\mathbf{v}_T \delta v_i^2 \rangle, \tag{46}$$

with $f(\theta) \equiv I(X)/X$ and

$$\begin{aligned} I(X) &= \int_0^X \sqrt{1+X^{1/3}}dX \\ &= -\frac{16}{35} + \frac{6}{7}(1+X^{1/3})^{3/2}X^{2/3} - \frac{24}{35}(1+X^{1/3})^{3/2}X^{1/3} \\ &\quad + \frac{16}{35}(1+X^{1/3})^{3/2}, \end{aligned} \tag{47}$$

$$X = \left(\frac{4}{3}\tan\,\theta\right)^6, \tag{48}$$

$$\mathbf{e_T} = \frac{\mathbf{e}^{(2)} + (4/3)\tan\,\theta\boldsymbol{\lambda}}{\sqrt{1+(4/3)^2\tan^2\,\theta}}. \tag{49}$$

It is the second main result of the paper. Several comments have to be made. First, the exact vectorial law for rotating turbulence has a form close to the isotropic case [1] with a scaling still linear in ρ. However, we observe a θ-angle dependence which reduces the degree of universality of the law. From an observational point of view this prediction turns out to be interesting since the measurements may be made at a given angle for which the law keeps a degree of universality. In Fig. 2 we give the evolution of the constant (up to the sign) according to the angle θ. We see a slight variation from 2 to 16/7 for, respectively, $\theta=0$ to $\pi/2$. Note that this variation corresponds to a fixed distance r. Second, we note that the vectorial nature of the law implies different components of the fluctuating fields with mainly the λ components for a large angle. Third, like for isotropic turbulence we have a contribution of all three components of the fluctuating field through scalar products. Fourth, we see that the exact relation implies only the energy dissipation rate per unit mass and *not* specific energy dissipation rate per unit mass for each direction, like $\varepsilon_\perp$ and $\varepsilon_\parallel$. We have here a difference with other types of exact results like in wave turbulence where the final result is expressed in terms of directional energy transfer fluxes (see, e.g., [23,33,34]). Fifth, the exact vectorial law is derived by assuming the existence of an external rotating rate. The extension of this law to a local

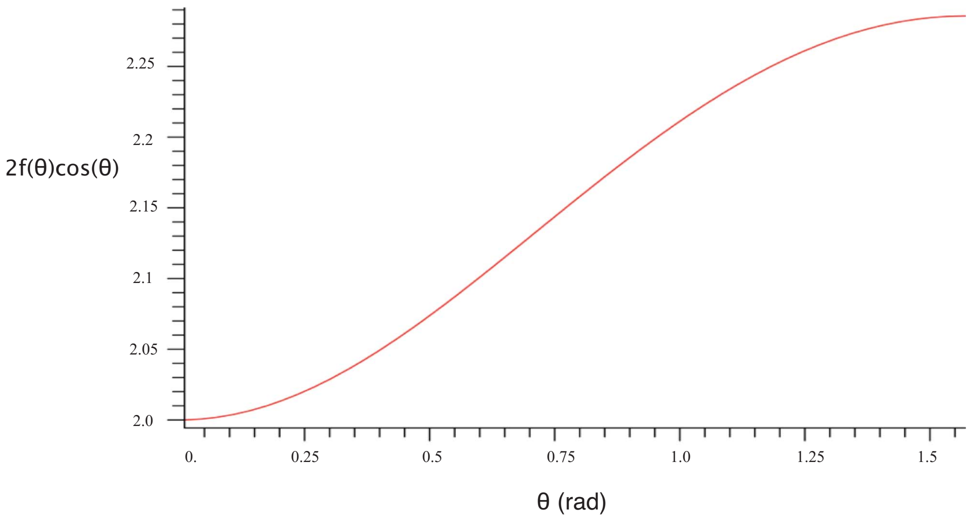

FIG. 2. (Color online) Variation of $2f(\theta)\cos\theta$ as a function of θ for $\theta \in [0,\pi/2]$. A slight variation is found going from 2 to 16/7.

analysis for which anisotropy is due to a local rotation might also be considered but then it is only an approximate law since in our derivation we have considered the entire inertial range.

1. Exact law for mainly radial correlations

The first interesting limit to analyze is the one for which the correlations are mainly radial, i.e., for small θ. In the limit of small angle, we obtain after a Taylor expansion

$$I(X) \simeq X + \frac{3}{8}X^{4/3}, \tag{50}$$

and then after substitution

$$-2\left(1+\frac{2}{3}\tan^2\theta\right)\varepsilon\rho\mathbf{e_T} \simeq \langle \delta\mathbf{v}_T \delta v_i^2\rangle. \tag{51}$$

We see that the scaling prediction for 2D turbulence (see Sec. IV B) is recovered at first order when the limit of radial correlations is taken since then the T components are mainly the radial components. However, there is big difference with the 2D case since here we still have a direct transfer and a negative sign.

2. Exact law for mainly axial correlations

The second interesting limit for which the exact vectorial law simplifies is the one for which the correlations are mainly axial, i.e., for large (close to $\pi/2$) θ angle. In the limit of large angle, we obtain after expansion

$$I(X) \simeq \frac{6}{7}X^{7/6} - \frac{24}{35}X^{5/6}, \tag{52}$$

and then after substitution

$$-\frac{16}{7}\left(1-\frac{9}{20}\frac{1}{\tan^2\theta}\right)\varepsilon z\mathbf{e_T} \simeq \langle \delta\mathbf{v}_T \delta v_i^2\rangle. \tag{53}$$

Note the dependence in z instead of ρ. In this case, the T-components mean mainly the axial components.

3. Relation between radial and axial correlations

From the previous relations (51) and (53) it is straightforward to derive (at first order) a relation between the radial (ρ) and axial (z) correlations

$$\varepsilon \simeq -\frac{1}{2}\frac{\langle \delta v_\rho \delta v_i^2\rangle}{\rho} \simeq -\frac{7}{16}\frac{\langle \delta v_z \delta v_i^2\rangle}{z}. \tag{54}$$

This relation may be seen as a proxy to measure the local dissipation.

V. DISCUSSION AND CONCLUSION

A. Anisotropic energy spectra

This work is mainly devoted to scaling laws in the physical space but it is tempting to discuss the consequences in Fourier space of the new exact vectorial law. Then it will be possible to compare our prediction to heuristic models for the anisotropic energy spectrum.

From a straightforward dimensional analysis we may predict the energy spectrum; we have $v_r^3 \sim \varepsilon f(\theta)\rho$ which gives (naively) the one-dimensional energy spectrum

$$E(k_r) = C_K(\theta)\varepsilon^{2/3}k_\perp^{-5/3}, \tag{55}$$

where by definition the wave number k_r is the one along $\mathbf{r}$, $k_\perp$ is a radial wave number, and $C_K(\theta)$ is a function of the angle θ which generalizes the Kolmogorov constant. Several comments have to be made. First, spectrum (55) is a one-dimensional spectrum different from the two- or three-dimensional one often used in anisotropic turbulence (see, e.g., [23,24]); for comparison we recall the relation $k_\perp = k_r \cos\theta$. Second, expression (55) is compatible with a classical Kolmogorov phenomenology where the inertial wave time, $\tau_I \sim k_r/k_z\Omega$, is not introduced. This is coherent with the vKH equation in the sense that the introduction of the Coriolis force does not change the form of Eq. (12) since the second-order correlation tensors proportional to Ω annihilate by symmetry. However, the presence of Ω—and thus the existence of an inertial wave time—is crucial for the geometric interpretation in terms of transfer direction: actually the dependence in θ of the function C_K is directly linked to rotation since it traduces the degree of anisotropy. Third, spectrum (55) does not seem to be compatible with steeper one-dimensional spectra generally found in the literature with an index around −2. Of course, the cause is possibly linked to the naive translation of the exact vectorial law into energy spectrum; it is better to use first the prediction in physical space to compare the theory with data. However, it is interesting to note that expression (51)—valid for small (but not necessarily fixed) θ angle—gives the following first-order correction to the scaling $v_r^3 \sim \rho^{5/3}$ which corresponds to the one-dimensional spectrum correction

$$\delta E \sim k_\perp^{-19/9}. \tag{56}$$

We see that this correction goes in the right direction with a power law steeper than the Komogorov one.

B. Conclusion

In this paper, I develop a theory for homogeneous weak axisymmetric turbulence. I show that the problem is solvable and an exact vectorial law is derived. The fundamental remark to achieve this goal is first to follow a formalism different from the one developed by Batchelor [10] or Chandrasekhar [11], and second to perform an integration over a manifold that corresponds to a scaling relationship between the axial and the radial components of the vector separation **r**. The general law is discussed in the case of rotating turbulence. The vectorial nature of the relation traduces the fact that we have two degrees of freedom (r and θ) instead of one (r) in the isotropic situation, and the angular dependence is intimately linked to the strength of Ω and the scale itself. The difference between the exact vectorial law derived for homogeneous rotating turbulence and the isotropic law resides in the prefactor and the T direction which is not the direction along the vector separation **r**.

The scenario proposed does not lead to any inverse cascade as often evoked to explain experimental results [18] or direct numerical simulations [20]. It is likely that the choice of manifolds prevents this scenario by selecting among the possible solutions of Eq. (27) the one corresponding to a direct cascade. Note that kinetic helicity is not included in the present analysis although it may have an important role in the dynamics of rotating turbulence. This point is currently under investigation and will be presented in a forthcoming paper.

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APPENDIX: KINEMATICS FOR HOMOGENEOUS, WEAK AXISYMMETRIC TURBULENCE

In this appendix we recall shortly the main results of the kinematics for homogeneous, weak axisymmetric turbulence useful for the discussion in the present paper. Other details are given in Lindborg [12].

1. Useful identities

We first recall the fundamental (trivial) identities that are useful for the development of the kinematics

$$\delta_{ij} = \lambda_i\lambda_j + e_i^{(1)}e_j^{(1)} + e_i^{(2)}e_j^{(2)}, \tag{A1}$$

$$r_i = r\sqrt{1 - z^2/r^2}\,e_i^{(2)} + z\lambda_i. \tag{A2}$$

Second, we give the (less trivial) relations on the derivative

$$\frac{\partial e_i^{(1)}}{\partial r_j} = -\frac{1}{\rho}e_i^{(2)}e_j^{(1)}, \tag{A3}$$

$$\frac{\partial e_i^{(2)}}{\partial r_j} = \frac{1}{\rho}e_i^{(1)}e_j^{(1)}, \tag{A4}$$

$$\lambda_j\frac{\partial}{\partial r_j} = \frac{\partial}{\partial z}, \tag{A5}$$

$$e_j^{(2)}\frac{\partial}{\partial r_j} = \frac{\partial}{\partial \rho}, \tag{A6}$$

$$e_j^{(1)}\frac{\partial}{\partial r_j} = 0. \tag{A7}$$

Note that the last three relations are valid if the operator acts on functions of ρ and z (and not on the basis vectors).

2. Linear form

From the three unit vectors $\boldsymbol{\lambda}$, $\mathbf{e}^{(1)}$, and $\mathbf{e}^{(2)}$ we may form three independent first-order tensors, each of which can be multiplied by a scalar function; we obtain

$$P_i(\rho,z) = P_1\lambda_i + P_2e_i^{(2)} + P_3e_i^{(1)}, \tag{A8}$$

where P_1, P_2, and P_3 are arbitrary functions of ρ and $z = \mathbf{r}\cdot\boldsymbol{\lambda}$. In practice, the first-order axisymmetric tensor will be associated to the total pressure-velocity correlation $P_i(\rho,z) = \langle Pv_i'\rangle$, with the pressure taken at point M and the velocity at point M'. It is a solenoidal tensor satisfying the relation $\partial P_i/\partial r_i = 0$; it is equivalent to

$$\frac{\partial(\rho P_2)}{\partial \rho} = -\rho\frac{\partial P_1}{\partial z}, \tag{A9}$$

which means that, contrary to the isotropic case, first-order solenoidal tensors are nonequal to zero.

3. Bilinear form

From the three unit vectors $\boldsymbol{\lambda}$, $\mathbf{e}^{(1)}$, and $\mathbf{e}^{(2)}$ we may form nine independent second-order tensors, each of which can be multiplied by a scalar function; we obtain

$$R_{ij}(\rho,z) = R_1\lambda_i\lambda_j + R_2e_i^{(2)}e_j^{(2)} + R_3e_i^{(1)}e_j^{(1)} + R_4(\lambda_ie_j^{(2)} + e_i^{(2)}\lambda_j) + R_6(\lambda_ie_j^{(1)} + e_i^{(1)}\lambda_j) + R_8(e_i^{(2)}e_j^{(1)} + e_i^{(1)}e_j^{(2)}), \tag{A10}$$

where R_n are arbitrary functions of ρ and z. Note the use of the index symmetry condition, $R_{ij}(\mathbf{r})=R_{ji}(-\mathbf{r})$, applicable for homogeneous turbulence (with the same type of fields) to reduce from nine to six the number of terms. In practice, this tensor will be associated with the velocity such that $R_{ij}(\rho,z)=\langle v_i v_j' \rangle$ which satisfies the continuity condition $\partial R_{ij}/\partial r_j=0$. Finally note the reduction for the diagonal part to $R_{ii}=R_1+R_2+R_3$.

4. Trilinear form

From the three unit vectors $\boldsymbol{\lambda}$, $\mathbf{e^{(1)}}$, and $\mathbf{e^{(2)}}$ we may form 27 independent triple-order tensors, each of which can be multiplied by a scalar function. For tensors symmetric in its first two indices we obtain the following simplified form:

$$S_{ikj}(\rho,z) = S_1\lambda_i\lambda_k\lambda_j + S_2 e_i^{(2)}e_k^{(2)}\lambda_j + S_3 e_i^{(1)}e_k^{(1)}\lambda_j + S_4\lambda_i\lambda_k e_j^{(2)} + S_5 e_i^{(2)}e_k^{(2)}e_j^{(2)} + S_6 e_i^{(1)}e_k^{(1)}e_j^{(2)} + S_7(\lambda_i e_k^{(2)} + e_i^{(2)}\lambda_k)\lambda_j + S_8(\lambda_i e_k^{(2)} + e_i^{(2)}\lambda_k)e_j^{(2)} + S_9(\lambda_i e_k^{(1)} + e_i^{(1)}\lambda_k)e_j^{(1)} + S_{10}(e_i^{(2)}e_k^{(1)} + e_i^{(1)}e_k^{(2)})e_j^{(1)} + S_{11}\lambda_i\lambda_k e_j^{(1)} + S_{12} e_i^{(2)}e_k^{(2)}e_j^{(1)} + S_{13} e_i^{(1)}e_k^{(1)}e_j^{(1)} + S_{14}(\lambda_i e_k^{(1)} + e_i^{(1)}\lambda_k)\lambda_j + S_{15}(\lambda_i e_k^{(1)} + e_i^{(1)}\lambda_k)e_j^{(2)} + S_{16}(e_i^{(2)}e_k^{(1)} + e_i^{(1)}e_k^{(2)})\lambda_j + S_{17}(\lambda_i e_k^{(2)} + e_i^{(2)}\lambda_k)e_j^{(1)} + S_{18}(e_i^{(2)}e_k^{(1)} + e_i^{(1)}e_k^{(2)})e_j^{(2)}, \tag{A11}$$

where S_n are arbitrary functions of ρ and z. In practice, the third-order tensor will be associated with $S_{i\ell j}(\mathbf{r})=\langle v_i v_\ell u_j' \rangle$. Note the following simplification after contraction of indices:

$$S_{iki} = (S_1 + S_8 + S_9)\lambda_k + (S_5 + S_7 + S_{10})e_k^{(2)} + (S_{13} + S_{14} + S_{18})e_k^{(1)}. \tag{A12}$$

This tensor satisfies also the divergence free condition

$$\frac{\partial S_{ikj}}{\partial r_j} = 0. \tag{A13}$$

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